University of California, Berkeley
Physics 105 Fall 2000 Section 2 (*Strovink*)

ASSIGNMENT 1

Reading:
105 Notes 1.1, 1.2, 1.3, 1.4, 1.5.
Hand & Finch 7.1, 7.2, 7.3, 7.4, and 8.7 (pp. 300-302 only).

1. A matrix A is called *orthogonal* if

$$A^{-1} = A^t \ ,$$

where

$$\left(A^t\right)_{ij} \equiv A_{ji} \ .$$

(a) Prove that the product of two orthogonal matrices is also orthogonal.
(b) Show that if A is a 3×3 orthogonal matrix, its three column vectors are mutually perpendicular and of unit length.

2. Suppose that a vector $\mathbf{x}'$ in the space axes is related to a vector $\mathbf{x}$ in the body axes by

$$\mathbf{x}' = A\mathbf{x} \ ,$$

where A is a transformation matrix. Given a matrix F, find a matrix F', expressed in terms of F and A, such that

$$\mathbf{x}'^t F' \mathbf{x}' = \mathbf{x}^t F \mathbf{x} \ .$$

F and F' are said to be related by a *similarity transformation.*

3. Define the *trace* of a matrix F as

$$\mathrm{Tr}(F) = F_{ij}\delta_{ij} \ ,$$

where, as usual, summation over repeated indices is implied.
(a) Show that $\mathrm{Tr}(F)$ is the sum of the diagonal elements of F.
(b) Prove that $\mathrm{Tr}(F)$ is invariant under any similarity transformation.

4.
(a) Use the Levi-Civita density ϵ_{ijk} to prove the *bac cab rule*

$$\mathbf{a} \times (\mathbf{b} \times \mathbf{c}) = \mathbf{b}(\mathbf{a} \cdot \mathbf{c}) - \mathbf{c}(\mathbf{a} \cdot \mathbf{b}) \ .$$

(b) Use the bac cab rule to show that

$$\mathbf{a} = \hat{\mathbf{n}}(\mathbf{a} \cdot \hat{\mathbf{n}}) + \hat{\mathbf{n}} \times (\mathbf{a} \times \hat{\mathbf{n}}) \ ,$$

where $\hat{\mathbf{n}}$ is any unit vector. What is the geometrical significance of each of the two terms in the expansion?

5. Consider three vectors $\mathbf{u}$, $\mathbf{v}$, and $\mathbf{w}$.
(a) Show that

$$(\mathbf{u} \times \mathbf{v}) \cdot \mathbf{w} = \epsilon_{ijk} u_i v_j w_k \ ,$$

where, as usual, summation is implied.
(b) If $\mathbf{u}$, $\mathbf{v}$, and $\mathbf{w}$ emanate from a common point, show that $|\epsilon_{ijk} u_i v_j w_k|$ is the volume of the parallelepiped whose edges they determine.

6. In a complex vector space, a matrix U is called *unitary* if

$$U^{-1} = U^\dagger \ ,$$

where

$$\left(U^\dagger\right)_{ij} \equiv U^*_{ji} \ .$$

Show than an *infinitesimal* unitary transformation T (one that is infinitesimally different from the unit matrix) can be written

$$T \approx I + iH \ ,$$

where I is the unit matrix and H is *Hermitian*, *i.e.*

$$H = H^\dagger \ .$$

7. Show that $\mathbf{v}$, $\mathbf{p}$, and $\mathbf{E}$ (velocity, momentum, and electric field) are ordinary ("polar") vectors, while $\boldsymbol{\omega}$, $\mathbf{L}$, and $\mathbf{B}$ (angular velocity, angular momentum, and magnetic field) are pseudo ("axial") vectors.

8. Find the transformation matrix Λ, such that

$$x'_i = \Lambda_{ij} x_j \ ,$$

which describes the following (passive) transformation: relative to the space (primed) axes, the body (unprimed) axes are rotated counterclockwise by an angle ξ about a unit vector $\hat{\mathbf{n}}'$ which has direction cosines n'_1, n'_2, and 0 with respect to the x'_1, x'_2, and x'_3 (space) axes, respectively.